Correcting Newton-Côtes integrals by Lévy areas

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Abstract

In this note we introduce the notion of Newton-Côtes integral corrected by Lévy areas, which enables us to consider integrals of the type $\int f(y)dx$, where f is a $\mathcal{C}^{2m}$ function and x, y are real Hölderian functions with index $\alpha > 1/(2m+1)$, for any $m \in \mathbb{N}^*$. We show that this concept extends the Newton-Côtes integral introduced in [8], to a larger class of integrands. Then, we give a theorem of existence and uniqueness for differential equations driven by x, interpreted using this new integral.

Key words: Fractional Brownian motion - Lévy area - Newton-Côtes integral - Rough differential equation.

MSC 2000: 60G18, 60H10

1 Introduction

Recent applications of stochastic processes are based on a modelling with differential equations driven by a fractional Brownian motion (fBm in short) B^H, of the type

$$X_t \;=\; x_0 \;+\; \int_0^t b(X_s)ds \;+\; \int_0^t \sigma(X_s)dB_s^H, \quad t \in [0,1], \tag{1.1}$$

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where $X = \{X_t,\ t \in [0,1]\}$ is the unknown continuous process and $x_0 \in \mathbb{R}$ and $b, \sigma : \mathbb{R} \to \mathbb{R}$ are the given data, see e.g. [3, 5] and the references therein. It is well-known that the fBm B^H of Hurst index $H \in (0,1)$ is a semimartingale if and only if $H = 1/2$, that is when it is the standard Brownian motion. Then, for $H \neq 1/2$, the sense of $\int_0^t \sigma(X_s)dB_s^H$ in (1.1) is not clear and has to be precised. Let us make a short recall of the three theories of integration with respect to fBm which are nowadays frequently used.

(a) In Russo-Vallois' theory [18], the (symmetric) integral is defined by

$$\int_0^t Z_s d^\circ B_s^H = \lim_{\varepsilon \to 0} -\text{ucp} \quad \varepsilon^{-1} \int_0^t \frac{Z_{s+\varepsilon} + Z_s}{2} (B_{s+\varepsilon}^H - B_s^H)\, ds, \tag{1.2}$$

provided the limit exists. When the integrand Z is of the type $Z_s = f(B_s^H)$, recent results - see [2, 8] - show that $\int_0^t f(B_s^H)d^\circ B_s^H$ exists for all regular enough functions $f : \mathbb{R} \to \mathbb{R}$ if and only if $H > 1/6$. When $Z_s = h(B_s^H, V_s)$ with V a process of bounded variation and $h : \mathbb{R}^2 \to \mathbb{R}$ a regular function, it was shown in [13] that $\int_0^t h(B_s^H, V_s)d^\circ B_s^H$ exists if $H > 1/3$. When $H \leq 1/3$, one can extend the definition (1.2) and give a sense to

$$\int_0^t h(B_s^H, V_s)dB_s^H \tag{1.3}$$

with the help of the m-order Newton-Côtes integral, which was introduced in [8] - see Definition 2.1 thereafter. Choosing m sufficiently large exhibits a stochastic integral which makes sense to (1.1) for *any* $H \in (0,1)$ [13]. However, one needs to suppose somewhat arbitrarily that the solution to (1.1) is *a priori* of the type $f(B_s^H, V_s)$.

(b) Another formalism relies upon the Malliavin calculus for fBm, in the sense of Nualart-Zakai [15], and more specifically on Skorohod's integration operator δ^H. Combining this with techniques of fractional calculus and Young integrals, one can then study (1.1) for $H > 1/2$ in any dimension - see [17], and also Nualart's survey article [16] for other topics of this theory.

(c) Finally, one can make a sense to (1.1) with the help of Lyons' theory of rough paths [9]. Roughly speaking, the goal of this theory is to give sense to quantities such as $\int_\gamma \omega$, where ω is a differential 1-form and γ a curve having only Hölder continuous regularity. In order to use it, it is then necessary to reinterpret (1.1) using a differential 1-form, through the formulation

$$X_t = x_0 + \int_{\gamma([0,t])} \omega \tag{1.4}$$

with $\gamma_t = (B_t^H, t, X_t) \in \mathbb{R}^3$ and $\omega = \sigma(x_3)dx_1 + b(x_3)dx_2$. Recent results [4, 7] establish that one can solve (1.4) only when $H > 1/4$, but in any dimension. Rough path theory has rich ramifications - see the monograph [10], but requires a formalism which is sometimes heavy.

It is quite natural to ask whether these different theories may intertwine with each other, and how. For instance, the following link is established between (a) and (b) in [1]: fixing a time-horizon T and $H > 1/2$, if u is a stochastic process such that

$$\int_0^T \int_0^T |D_s u_t||t-s|^{2H-2} ds dt \ < \ +\infty$$

and regular enough, then its symmetric integral along B^H exists and is given by

$$\int_0^T u_t d^\circ B_t^H \ = \ \delta^H(u) \ + \ c_H \int_0^T \int_0^T D_s u_t |t-s|^{2H-2} ds dt,$$

where D_s stands for the Malliavin derivative and δ^H for the Skorohod integral [15]. The present note wishes to link (a) and (c). We propose a correction of the Newton-Côtes integral $d^{\mathrm{NC},m}$ by some *Lévy areas*, which are the central object in rough paths' theory. Our new integrator $d^{A,m}$ defines, for any $m \in \mathbb{N}^*$

$$\int_0^t f(y_s) d^{A,m} x_s$$

when $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^m$ and x, y are any fractal functions of index $\alpha > 1/(2m+1)$ (Theorem 2.5). Compared to a) our class of integrands is much more satisfactory, because y need not depend on x anymore. Compared to b) and c) we reach a lower level for H, but a main drawback is that our approach is genuinely one-dimensional.

In the second part of the paper we prove existence and uniqueness for (1.1) driven by a fractal function of index $\alpha > 1/(2m+1)$ through our integral $d^{A,m}$, under some standard conditions on the coefficients (Theorem 3.2). The proof relies on Banach's fixed point theorem. Finally, we notice that for $m = 1$ and $y_t = g(x_t, \ell_t)$ with ℓ of bounded variation, one can choose a first order Lévy area such that the operators $d^{A,1}$ and d° actually coincide (Proposition 4.2). We are not sure whether an identification with Newton-Côtes integrals can be pursued for $m \geq 2$, because of the (crucial) Chasles relationship in the definition of Lévy areas.

This paper was mainly inspired by [7], more precisely by its first draft. For example, our constance lemma 2.7, which is key in establishing Theorem 2.5, can be viewed as a continuous analogue to the "sewing lemma" 2.1 therein. The possibility of reaching any value of H after considering families of Levy areas is also strongly suggested in [7]. However, our framework is continuous and in particular, our integrals are true integrals for $H > 1/3$, which may look more natural. Above all, we feel that this formalism is one of the simplest possible, and provides a handy framework for a more advanced stochastic analysis of (1.1), examples of which can be found in [11] and [14].

2 Newton-Côtes integrals corrected by Lévy areas

We fix once and for all $m \in \mathbb{N}^*$ and $\alpha \in (1/(2m+1), 1)$. We also consider, without loss of generality, functions which are defined on the interval $[0,1]$. Denote by C^α the set of fractal functions $z : [0,1] \to \mathbb{R}$ of index α, i.e. for which

$$\exists L > 0 \text{ such that } \forall s,t \in [0,1],\ |z_t - z_s| \leq L\,|t-s|^\alpha. \tag{2.5}$$

Introduce the interpolation measure ν_m given by

$$\nu_1 = \frac{1}{2}(\delta_0 + \delta_1) \qquad \text{if } m = 1,$$

$$\nu_m = \sum_{j=0}^{2m-2} \left(\int_0^1 \left(\prod_{k \neq j} \frac{(2(m-1)u - k)}{j-k} \right) du \right) \delta_{j/(2m-2)} \qquad \text{if } m \geq 2,$$

where δ stands for the Dirac mass. This measure is the unique discrete measure carried by the numbers $j/(2m-2)$ which coincides with Lebesgue measure on polynoms of degree smaller than $2m-1$. In [8], the Newton-Côtes integral was defined followingly:

Definition 2.1. *Let $x : [0,1] \to \mathbb{R}$, $z : [0,1] \to \mathbb{R}^2$ and $h : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions. The integral defined by:*

$$\int_0^t h(z_s) d^{\mathrm{NC},m} x_s \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^t ds\, (x_{s+\varepsilon} - x_s) \int_0^1 h((1-\alpha)z_s + \alpha z_{s+\varepsilon}) \nu_m(d\alpha) \tag{2.6}$$

provided the limit exists, is called the m-order Newton-Côtes integral $I_m(h,z,x)$ *of $h(z)$ with respect to x.*

Remarks 2.2. (a) When $m = 1$, Newton-Côtes integral is a true integral which coincides with the symmetric integral $\int_0^t h(z_s) d^\circ x_s$ given in Definition (1.2).

(b) When $m \geq 2$, Newton-Côtes integral is not a true integral anymore since if $h(z) = \tilde{h}(\tilde{z})$, the identification

$$\int_0^T h(z_s) d^{\mathrm{NC},m} x_s = \int_0^T \tilde{h}(\tilde{z}_s) d^{\mathrm{NC},m} x_s$$

does not hold in general.

Notice that there is no reason *a priori* that the integral $I_m(h,z,x)$ exists. In [13], this was established when z is of the form $u \mapsto f(x_u, \ell_u)$ where $\ell : [0,1] \to \mathbb{R}$ has bounded variations and $f : \mathbb{R}^2 \to \mathbb{R}$ is regular enough. In order to extend the class of integrands, we wish to define a new concept of integral. To do so, let us first define the notion of Lévy area:

Definition 2.3. *Let $x, y : [0,1] \to \mathbb{R}$ be two functions belonging to* C^α *and $\gamma : [0,1] \to \mathbb{R}^2$ be the curve given by $\gamma_t = (x_t, y_t)$. If $r, s, t \in [0,1]$, we denote by T_{rst} the oriented triangle with vertices γ_r, γ_s and γ_t. We say that A is* a Lévy area of order $2m-2$ associated to *γ if $\forall s, t \in [0,1]$ $P \to A_{st}(P)$ is a linear map from $\mathcal{P}_{2m-2}$ (the space of polynomials in y with degree $\leq 2m-2$) into* $\mathbb{R}$, *if* $\forall r, s, t \in [0,1]$, $\forall k \in \{0, \ldots, 2m-2\}$,

$$A_{rs}(y^k) + A_{st}(y^k) + A_{tr}(y^k) = -\iint_{T_{rst}} y^k dxdy \tag{2.7}$$

and if $\exists c > 0$ *s.t.* $\forall s, t \in [0,1]$, $\forall k \in \{0, \ldots, 2m-2\}$, $\forall \xi \in [y_s, y_t]$:

$$|A_{st}[(y-\xi)^k]| \ \leq \ c|t-s|^{2m\alpha}. \tag{2.8}$$

Remark 2.4. From (2.8), we see that $A_{ss}(P) = 0$ for any $s \in [0,1]$ and $P \in \mathcal{P}_{2m-2}$. From (2.7) and since $\iint_{T_{sst}} y^k dxdy = 0$, we see that $A_{st}(P) = -A_{ts}(P)$ for any $s, t \in [0,1]$ and $P \in \mathcal{P}_{2m-2}$.

We can now give the main result and the central definition of this paper:

Theorem 2.5. *Let $x, y \in \mathrm{C}^\alpha$ with $\alpha > 1/(2m+1)$ and A be a Lévy area of order $2m-2$ associated to $\gamma = (x, y)$. For $f : \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^{2m}$-function, define*

$$\begin{aligned} I_\varepsilon^\gamma(f) \ &= \ \varepsilon^{-1} \int_0^1 du(x_{u+\varepsilon} - x_u) \int_0^1 f((1-\alpha)y_u + \alpha y_{u+\varepsilon})\nu_m(d\alpha) \\ &\quad + \ \varepsilon^{-1} \sum_{k=0}^{2m-2} \frac{1}{(k+1)!} \int_0^1 f^{(k+1)}(y_u) A_{u,u+\varepsilon}[(y-y_u)^k] du \end{aligned}$$

for every $\varepsilon > 0$. Then the family $\{I_\varepsilon^\gamma(f),\, \varepsilon > 0\}$ converges when $\varepsilon \downarrow 0$. Its limit is denoted

$$I^\gamma(f) \ = \ \int_0^1 f(y_u) d^{A,m} x_u$$

and is called the Newton-Côtes integral corrected by A of $f(y)$ with respect to x*.*

The proof of Theorem 2.5 relies upon the two following lemmas. We fix $f \in \mathcal{C}^{2m}(\mathbb{R}, \mathbb{R})$ once and for all.

Lemma 2.6. *Set*

$$\begin{aligned} I_n(\varepsilon) \ &= \ 2^n \varepsilon^{-1} \int_0^{\varepsilon\left[\frac{1}{\varepsilon}\right]} du(x_{u+\varepsilon 2^{-n}} - x_u) \int_0^1 f((1-\alpha)y_u + \alpha y_{u+\varepsilon 2^{-n}})\nu_m(d\alpha) \\ &\quad + \ 2^n \varepsilon^{-1} \sum_{k=0}^{2m-2} \frac{1}{(k+1)!} \int_0^{\varepsilon\left[\frac{1}{\varepsilon}\right]} f^{(k+1)}(y_u) A_{u,u+\varepsilon 2^{-n}}[(y-y_u)^k] du \end{aligned}$$

for every $\varepsilon > 0$ and $n \in \mathbb{N}$. The sequence of functions $\{I_n,\, n \in \mathbb{N}\}$ converges uniformly on each compact of $]0,1]$, and the limit I_∞ verifies

$$I_\infty(\varepsilon) \ = \ I_\varepsilon^\gamma(f) \ + \ O(\varepsilon^{[(2m+1)\alpha - 1]\wedge\alpha}). \tag{2.9}$$

Proof. First, assume that $m=1$. In this case, we have

$$I_n(\varepsilon) \;=\; 2^n\varepsilon^{-1}\left(\int_0^{\varepsilon\left[\frac{1}{\varepsilon}\right]}\frac{f(y_u)+f(y_{u+\varepsilon 2^{-n}})}{2}(x_{u+\varepsilon 2^{-n}}-x_u)du\;+\;\int_0^{\varepsilon\left[\frac{1}{\varepsilon}\right]}f'(y_u)A_{u,u+\varepsilon 2^{-n}}du\right),$$

where, for the simplicity of the exposition, we wrote A_{st} instead of $A_{st}(1)$. Decomposing the integral into dyadic intervals and making a change of variable, we first get

$$\begin{aligned}
I_n(\varepsilon) \;&=\; 2^n\varepsilon^{-1}\sum_{k=0}^{\left[\frac{1}{\varepsilon}\right]2^n-1}\int_{\varepsilon k2^{-n}}^{\varepsilon(k+1)2^{-n}}\left[\frac{f(y_u)+f(y_{u+\varepsilon 2^{-n}})}{2}(x_{u+\varepsilon 2^{-n}}-x_u)+f'(y_u)A_{u,u+\varepsilon 2^{-n}}\right]du\\
&=\;\sum_{k=0}^{\left[\frac{1}{\varepsilon}\right]2^n-1}\int_0^1\left[\frac{f(y^n_k)+f(y^n_{k+1})}{2}\left(x^n_{k+1}-x^n_k\right)\;+\;f'(y^n_k)A^n_{k,k+1}\right]du.
\end{aligned}$$

where, for simplicity of exposition, we wrote $x^n_k=x_{\varepsilon 2^{-n}(k+u)}$, $y^n_k=y_{\varepsilon 2^{-n}(k+u)}$ and $A^n_{k,\ell}=A_{\varepsilon 2^{-n}(k+u),\varepsilon 2^{-n}(\ell+u)}$. Dividing again in two, we find

$$\begin{aligned}
I_{n+1}(\varepsilon) \;&=\;\sum_{k=0}^{\left[\frac{1}{\varepsilon}\right]2^n-1}\int_0^1\left[\frac{f(y^{n+1}_{2k})+f(y^{n+1}_{2k+1})}{2}\left(x^{n+1}_{2k+1}-x^{n+1}_{2k}\right)\;+\;f'(y^{n+1}_{2k})A^{n+1}_{2k,2k+1}\right]du\\
&+\;\sum_{k=0}^{\left[\frac{1}{\varepsilon}\right]2^n-1}\int_0^1\left[\frac{f(y^{n+1}_{2k+1})+f(y^{n+1}_{2k+2})}{2}\left(x^{n+1}_{2k+2}-x^{n+1}_{2k+1}\right)\;+\;f'(y^{n+1}_{2k+1})A^{n+1}_{2k+1,2k+2}\right]du.
\end{aligned}$$

On the other hand, after another change of variable, we can rewrite

$$\begin{aligned}
I_n(\varepsilon) \;&=\;\frac{1}{2}\sum_{k=0}^{\left[\frac{1}{\varepsilon}\right]2^n-1}\int_0^1\left[\frac{f(y^{n+1}_{2k})+f(y^{n+1}_{2k+2})}{2}\left(x^{n+1}_{2k+2}-x^{n+1}_{2k}\right)\;+\;f'(y^{n+1}_{2k})A^{n+1}_{2k,2k+2}\right]du\\
&+\;\frac{1}{2}\sum_{k=0}^{\left[\frac{1}{\varepsilon}\right]2^n-1}\int_0^1\left[\frac{f(y^{n+1}_{2k+1})+f(y^{n+1}_{2k+3})}{2}\left(x^{n+1}_{2k+3}-x^{n+1}_{2k+1}\right)\;+\;f'(y^{n+1}_{2k+1})A^{n+1}_{2k+1,2k+3}\right]du.
\end{aligned}$$

Writing $J_n(\varepsilon)=I_{n+1}(\varepsilon)-I_n(\varepsilon)$, this yields

$$\begin{aligned}
J_n(\varepsilon) \;&=\;\frac{1}{2}\sum_{k=0}^{\left[\frac{1}{\varepsilon}\right]2^n-1}\int_0^1\Big[f'(y^{n+1}_{2k})A^{n+1}_{2k,2k+1}\;+\;f'(y^{n+1}_{2k+1})A^{n+1}_{2k+1,2k+2}\;-\;f'(y^{n+1}_{2k})A^{n+1}_{2k,2k+2}\\
&+\;\frac{f(y^{n+1}_{2k})+f(y^{n+1}_{2k+1})}{2}(x^{n+1}_{2k+1}-x^{n+1}_{2k})\;+\;\frac{f(y^{n+1}_{2k+1})+f(y^{n+1}_{2k+2})}{2}(x^{n+1}_{2k+2}-x^{n+1}_{2k+1})\\
&-\;\frac{f(y^{n+1}_{2k})+f(y^{n+1}_{2k+2})}{2}(x^{n+1}_{2k+2}-x^{n+1}_{2k})\Big]\,du\\
&+\;\frac{1}{2}\sum_{k=0}^{\left[\frac{1}{\varepsilon}\right]2^n-1}\int_0^1\Big[f'(y^{n+1}_{2k})A^{n+1}_{2k,2k+1}\;+\;f'(y^{n+1}_{2k+1})A^{n+1}_{2k+1,2k+2}\;-\;f'(y^{n+1}_{2k+1})A^{n+1}_{2k+1,2k+3}\\
&+\;\frac{f(y^{n+1}_{2k})+f(y^{n+1}_{2k+1})}{2}(x^{n+1}_{2k+1}-x^{n+1}_{2k})\;+\;\frac{f(y^{n+1}_{2k+1})+f(y^{n+1}_{2k+2})}{2}(x^{n+1}_{2k+2}-x^{n+1}_{2k+1})
\end{aligned}$$

$$
\begin{aligned}
&- \quad \frac{f(y_{2k+1}^{n+1}) + f(y_{2k+3}^{n+1})}{2}(x_{2k+3}^{n+1} - x_{2k+1}^{n+1})\Big] \, du \\
&= \quad \frac{1}{2} \sum_{k=0}^{[\frac{1}{\varepsilon}]2^n - 1} \int_0^1 \big[f'(y_{2k}^{n+1}) A_{2k,2k+1}^{n+1} \; + \; f'(y_{2k}^{n+1}) A_{2k+1,2k+2}^{n+1} \; + \; f'(y_{2k}^{n+1}) A_{2k+2,2k}^{n+1} \\
&+ \quad \frac{f(y_{2k}^{n+1}) + f(y_{2k+1}^{n+1})}{2}(x_{2k+1}^{n+1} - x_{2k}^{n+1}) \; + \; \frac{f(y_{2k+1}^{n+1}) + f(y_{2k+2}^{n+1})}{2}(x_{2k+2}^{n+1} - x_{2k+1}^{n+1}) \\
&- \quad \frac{f(y_{2k}^{n+1}) + f(y_{2k+2}^{n+1})}{2}(x_{2k+2}^{n+1} - x_{2k}^{n+1})\Big] \, du \; + \; O((\varepsilon 2^{-n})^{3\alpha - 1}) \\
&+ \quad \frac{1}{2} \sum_{k=0}^{[\frac{1}{\varepsilon}]2^n - 1} \int_0^1 \big[f'(y_{2k+1}^{n+1}) A_{2k+1,2k+2}^{n+1} \; + \; f'(y_{2k+1}^{n+1}) A_{2k+2,2k+3}^{n+1} \; - \; f'(y_{2k+1}^{n+1}) A_{2k+1,2k+3}^{n+1} \\
&+ \quad \frac{f(y_{2k+2}^{n+1}) + f(y_{2k+3}^{n+1})}{2}(x_{2k+3}^{n+1} - x_{2k+2}^{n+1}) \; + \; \frac{f(y_{2k+1}^{n+1}) + f(y_{2k+2}^{n+1})}{2}(x_{2k+2}^{n+1} - x_{2k+1}^{n+1}) \\
&- \quad \frac{f(y_{2k+1}^{n+1}) + f(y_{2k+3}^{n+1})}{2}(x_{2k+3}^{n+1} - x_{2k+1}^{n+1})\Big] \, du \; + \; O((\varepsilon 2^{-n})^{\alpha \wedge (3\alpha - 1)}) \\
&= \quad \frac{1}{2} \sum_{k=0}^{[\frac{1}{\varepsilon}]2^n - 1} \int_0^1 \Big[f'(y_{2k}^{n+1}) \mathcal{A}\left(T_{\gamma_{2k+2}\gamma_{2k+1}\gamma_{2k}}\right) \; + \; \frac{f(y_{2k+2}^{n+1}) - f(y_{2k+1}^{n+1})}{2}(x_{2k}^{n+1} - x_{2k+1}^{n+1}) \\
&- \quad \frac{f(y_{2k}^{n+1}) - f(y_{2k+1}^{n+1})}{2}(x_{2k+2}^{n+1} - x_{2k+1}^{n+1})\Big] \, du \\
&+ \quad \frac{1}{2} \sum_{k=0}^{[\frac{1}{\varepsilon}]2^n - 1} \int_0^1 \Big[f'(y_{2k+1}^{n+1}) \mathcal{A}\left(T_{\gamma_{2k+3}\gamma_{2k+2}\gamma_{2k+1}}\right) \; + \; \frac{f(y_{2k+3}^{n+1}) - f(y_{2k+2}^{n+1})}{2}(x_{2k+1}^{n+1} - x_{2k+2}^{n+1})\Big) \\
&- \quad \frac{f(y_{2k+1}^{n+1}) - f(y_{2k+2}^{n+1})}{2}(x_{2k+3}^{n+1} - x_{2k+2}^{n+1})\Big] \, du \; + \; O((\varepsilon 2^{-n})^{\alpha \wedge (3\alpha - 1)}) \\
&= \quad \frac{1}{2} \sum_{k=0}^{[\frac{1}{\varepsilon}]2^n - 1} \int_0^1 f'(y_{2k}^{n+1}) \Big[\mathcal{A}\left(T_{\gamma_{2k+2}\gamma_{2k+1}\gamma_{2k}}\right) \; + \; \frac{y_{2k+2}^{n+1} - y_{2k+1}^{n+1}}{2}(x_{2k}^{n+1} - x_{2k+1}^{n+1}) \\
&- \quad \frac{y_{2k}^{n+1} - y_{2k+1}^{n+1}}{2}(x_{2k+2}^{n+1} - x_{2k+1}^{n+1})\Big] \, du \\
&+ \quad \frac{1}{2} \sum_{k=0}^{[\frac{1}{\varepsilon}]2^n - 1} \int_0^1 f'(y_{2k+1}^{n+1}) \Big[\mathcal{A}\left(T_{\gamma_{2k+3}\gamma_{2k+2}\gamma_{2k+1}}\right) \; + \; \frac{y_{2k+3}^{n+1} - y_{2k+2}^{n+1}}{2}(x_{2k+1}^{n+1} - x_{2k+2}^{n+1}) \\
&- \quad \frac{y_{2k+1}^{n+1} - y_{2k+2}^{n+1}}{2}(x_{2k+3}^{n+1} - x_{2k+2}^{n+1})\Big] \, du \; + \; O((\varepsilon 2^{-n})^{\alpha \wedge (3\alpha - 1)}),
\end{aligned}
$$

where $\mathcal{A}\left(T_{abc}\right)$ stands for the oriented area of the triangle T_{abc}, and where the simplifications come from (2.8), the $\mathcal{C}^2$-regularity of f and the fact that x, y are $\alpha-$Hölder. Now since

$$\frac{1}{2}\left[(y_c - y_b)(x_a - x_b) - (y_a - y_b)(x_c - x_b)\right] \;=\; \mathcal{A}\left(T_{abc}\right), \tag{2.10}$$

we finally obtain

$$I_{n+1}(\varepsilon) - I_n(\varepsilon) \;=\; O((\varepsilon 2^{-n})^{\alpha\wedge(3\alpha-1)}),$$

which yields the desired uniform convergence of $\{I_n,\, n \in \mathbb{N}\}$ towards some I_∞. Besides, since $I_0(\varepsilon) = I_\varepsilon^\gamma(f)$, we have

$$I_\infty(\varepsilon) \;=\; I_\varepsilon^\gamma(f) \;+\; O(\varepsilon^{\alpha\wedge(3\alpha-1)}).$$

This completes the proof in the case $m = 1$. Let us explain briefly how it extends in the general case $m \geq 2$. Let Δ_n be the set of dyadics of order n on $[0, 1]$ and use the notation $t' = t + 2^{-n}$ and $\tau = \frac{t+t'}{2}$ for $t \in \Delta_n$. Let $\{w_n\}$ be the sequence defined by

$$\begin{aligned} w_n \;&=\; \sum_{t\in\Delta_n}(x_{t'} - x_t)\int_0^1 f((1-\alpha)y_t + \alpha y_{t'})\nu_m(d\alpha) \\ &\qquad\qquad + \sum_{k=0}^{2m-2}\frac{1}{(k+1)!}\sum_{t\in\Delta_n} f^{(k+1)}(y_t)A_{tt'}[(y-y_t)^k]. \end{aligned}$$

Using a Taylor expansion - see Lemma 6.2 in the first draft of [7], one can show that there exists a decomposition $w_{n+1} - w_n = U_n + V_n$ with $|U_n| \leq \mathrm{cst} 2^{n(1-(2m+1)\alpha)}$ and

$$V_n = \sum_{k=0}^{2m-2}\frac{1}{(k+1)!}\left(\sum_{t\in\Delta_n}\left\{f^{(k+1)}(y_\tau)A_{\tau t'}[(y-y_\tau)^k] - f^{(k+1)}(y_t)A_{tt'}[(y-y_t)^k]\right\}\right).$$

Hence, $|V_n| \leq \mathrm{cst} 2^{n(1-(2m+1)\alpha)}$ and the sequence $\{w_n\}$ converges absolutely. One can then finish the proof exactly as in the case $m = 1$.

□

Lemma 2.7. *The function I_∞ is constant on* $[0, 1]$.

Proof. As for the proof of Lemma 2.6, we only consider the case $m = 1$. The general case $m \geq 2$ can be handled analogously, with heavier notations. Once again, we set A_{st} for $A_{st}(1)$. It is clear from the definition of I_n and the unicity of the limit I_∞ that

$$I_\infty(1) = I_\infty(2^{-1}) = I_\infty(2^{-2}) = \cdots = I_\infty(2^{-n}) = \cdots \tag{2.11}$$

for all $n \in \mathbb{N}$. We next prove that I_∞ is constant on dyadics. From (2.11) and an induction argument, it suffices to prove that, if $k2^{-n}$ and $(k + 1)2^{-n}$ are two dyadics such that

$I_\infty(k2^{-n}) = I_\infty((k+1)2^{-n}) = \ell$, then $I_\infty((k+1/2)2^{-n}) = \ell$. Using the notation $k_n^m = k2^{-(n+m)}$ we have, for any $m \in \mathbb{N}$,

$$
\begin{aligned}
I_m((k+1/2)2^{-n}) &= \frac{2^{n+m}}{2k+1}\int_0^1 [f(y_u) + f(y_{u+(k+1/2)_n^m})](x_{u+(k+1/2)_n^m} - x_u)du \\
&+ \frac{2^{n+m+1}}{2k+1}\int_0^1 f'(y_u)A_{u,u+(k+1/2)_n^m}du \\
&= \frac{2^{n+m}}{2k+1}\int_0^1 [f(y_{u+1_{n+1}^m}) + f(y_{u+(k+1)_n^m})](x_{u+(k+1)_n^m} - x_{u+1_{n+1}^m})du \\
&+ \frac{2^{n+m}}{2k+1}\int_0^1 f'(y_{u+1_{n+1}^m})A_{u+1_{n+1}^m,u+(k+1)_n^m}du + O(2^{-m\alpha}) \\
&= \frac{2k+2}{2k+1}I_m((k+1)2^{-n}) - \frac{1}{2k+1}I_{n+m+1}(1) \; + \; O(2^{-m[(3\alpha-1)\wedge\alpha]}) \\
&+ \frac{2^{n+m}}{2k+1}\int_0^1 [f(y_{u+1_{n+1}^m}) - f(y_u)](x_{u+(k+1)_n^m} - x_{u+1_{n+1}^m})du \\
&- \frac{2^{n+m}}{2k+1}\int_0^1 [f(y_{u+(k+1)_n^m}) - f(y_{u+1_{n+1}^m})](x_{u+1_{n+1}^m} - x_u)du \\
&+ \frac{2^{n+m+1}}{2k+1}\int_0^1 f'(y_u)(A_{u+1_{n+1}^m,u+(k+1)_n^m} - A_{u,u+(k+1)_n^m} + A_{u,u+1_{n+1}^m})du \\
&= \frac{2k+2}{2k+1}I_m((k+1)2^{-n}) \; - \; \frac{1}{2k+1}I_{n+m+1}(1) \; + \; O(2^{-m[(3\alpha-1)\wedge\alpha]}),
\end{aligned}
$$

where the last line comes from (2.10). Making $m \to \infty$ yields

$$
I_\infty((k+1/2)2^{-n}) = \frac{2k+2}{2k+1}\ell - \frac{1}{2k+1}\ell = \ell,
$$

which proves that I_∞ is constant on the dyadics of $[0,1]$. Now since $I_n(\varepsilon)$ is obviously continuous in ε and since the convergence in Lemma 2.6 is uniform, Dini's lemma entails that $I_\infty(\varepsilon)$ is continuous. Hence, I_∞ is constant on $[0,1]$, as desired.

□

Proof of Theorem 2.5. From Lemma 2.7 and (2.9), we have

$$
I_\varepsilon^\gamma(f) \; = \; I_\infty(0) \; + \; O(\varepsilon^{((2m+1)\alpha-1)\wedge\alpha})
$$

which, since $\alpha > 1/(2m+1)$, proves the convergence of $\{I_\varepsilon^\gamma(f),\, \varepsilon > 0\}$ towards some limit I^γ.

□

3 Differential equations driven by fractal functions

Recent works study equation of type (1.1) in the Russo-Vallois setting and in a Stratonovich sense. For example, in [6], existence and uniqueness are proved for $H > 1/3$ with the following definition (see Definition 4.1 in [6]): a solution X to (1.1) is a process such that (X, B^H) is a symmetric vector cubic variation process (see Definition 3.12 in [6]) and such that for every smooth $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and every $t \geq 0$,

$$\int_0^t Z_s d^\circ X_s \;=\; \int_0^t Z_s b(X_s)ds \;+\; \int_0^t Z_s \sigma(X_s) d^\circ B_s^H \;-\; \frac{1}{4}\int_0^t \sigma\sigma'(X_s) d\left[Z, B^H, B^H\right]_s$$

where $Z_s = \varphi(X_s, B_s^H)$ and $\left[Z, B^H, B^H\right]_s$ is the cubic covariation defined in [6], p. 263. In [13] another type of equation is proposed, relying on the Newton-Côtes integrator and allowing to reach any value of H, but the solution is supposed a priori to be of the kind $X_s = f(B_s^H, V_s)$ with V of bounded variation.

In this section we present yet another approach which is more general and, hopefully, simpler. We work in the framework of fractal functions with index $\alpha > 1/(2m+1)$ and consider the formal equation

$$dy_t \;=\; b(y_t)dt \;+\; \sigma(y_t)dx_t \tag{3.12}$$

with $x \in \mathrm{C}^\alpha$. Fix $\alpha > 1/(2m+1)$ and a time-horizon $T = 1$ once and for all.

Definition 3.1. *A solution to (3.12) is a couple (y, A) verifying:*

- *$y : [0,1] \to \mathbb{R}$ belongs to C^α,*
- *A is a Lévy area of order $2m-2$ associated to (x, y),*
- *For any $t \in [0,1]$,*

$$y_t \;=\; y_0 \;+\; \int_0^t b(y_s)ds \;+\; \int_0^t \sigma(y_s) d^{A,m} x_s.$$

In this definition, we see that the sense which is given to

$$\int_0^t \sigma(y_s)dx_s$$

in (3.12) is contained in the concept of solution. The proof of the following theorem is a simple consequence of the Banach fixed point theorem and is mainly inspired by the first draft of [7].

Theorem 3.2. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^{2m}$−function and $b : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then (3.12) admits an unique solution (y, A).*

Proof. For simplicity we assume that $m = 1$ and $y_0 = 0$. The general case can be handled analogously. Consider E^α the set of couples (y, A) with $y : [0, 1] \to \mathbb{R}$ in C^α and A a Lévy area of order 0 associated to (x, y), endowed with the norm

$$N(y, A) = |y_0| + \sup_{t \neq s} \frac{|y_t - y_s|}{|t - s|^\alpha} + \sup_{t \neq s} \frac{|A_{ts}(1)|}{|t - s|^{2\alpha}} < \infty.$$

With this norm, one can show that E^α is a Banach space. Besides, for every $\delta > 0$, if E^α_δ denotes the set of restrictions of $(y, A) \in E^\alpha$ to $[0, \delta]$, then E^α_δ is also a Banach space endowed with the norm N. Considering $(y, A) \in E^\alpha$ and

$$\tilde{y}_t = \int_0^t \sigma(y_s) d^{A,1} x_s + \int_0^t b(y_s) ds, \quad t \in [0, 1],$$

and

$$\tilde{A}_{st}(1) = \int_s^t x_u \, \sigma(y_u) d^{A,1} x_u + \int_s^t x_u \, b(y_u) du - \frac{1}{2}(x_t + x_s)(\tilde{y}_t - \tilde{y}_s), \quad (s, t) \in [0, 1]^2,$$

it is not difficult to prove that $(\tilde{y}, \tilde{A}) \in E^\alpha$. Let $T : E^\alpha \to E^\alpha$ be defined by $T(y, A) = (\tilde{y}, \tilde{A})$ and $E^\alpha_\delta(R)$ be the set of couples $(y, A) \in E^\alpha_\delta$ verifying $N(y, A) \leq R$. Using the same arguments as in the proof of the first draft of [7], Theorem 11.4, we can show that there exists $R > 0$ sufficiently large and $\delta > 0$ sufficiently small such that T stabilizes and contracts $E^\alpha_\delta(R)$. Thanks to the Banach fixed point theorem, we deduce that T admits an unique fixed point $(y, a) \in E^\alpha_R(\delta)$. Since we can do the same thing on $[\delta, 2\delta], [2\delta, 3\delta] \ldots$ we obtain finally an unique solution (y, A) defined on $[0, 1]$.

□

4 The case of Russo-Vallois symmetric integral

In this section we show how the corrected symmetric integral (which corresponds to the case where $m = 1$) defined in Theorem 2.5 extends the Russo-Vallois symmetric integral, when the class of integrands is more specific. Here, we fix $\alpha \in (1/3, 1)$ once and for all.

Lemma 4.1. *Let $x : [0, 1] \to \mathbb{R}$ be a function in C^α, $h : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^{2,1}$-function and $\ell : [0, 1] \to \mathbb{R}$ be a function of bounded variation. Define $y : [0, 1] \to \mathbb{R}$ by $y_t = h(x_t, \ell_t)$. Then $y \in \mathrm{C}^\alpha$ and the Russo-Vallois symmetric integral $\int_r^s y d^\circ x$ exists $\forall r, s \in [0, 1]$. Moreover, the function A defined by*

$$A_{rs}(1) = \int_r^s y d^\circ x - \frac{y_r + y_s}{2}(x_s - x_r) \tag{4.13}$$

is a Lévy area of order 0 associated to $\gamma = (x, y)$, satisfying

$$\exists\, L > 0 \text{ such that } \forall r, s \in [0, 1],\ |A_{rs}(1)| \leq L|s - r|^{3\alpha}. \tag{4.14}$$

Proof. For simplicity we only consider the case $y_t = h(x_t)$ with $h : \mathbb{R} \to \mathbb{R}$ a $\mathcal{C}^2$-function. The general case can be proven analogously. The fact that $y \in \mathrm{C}^\alpha$ and that $\int_r^s h(x) d^\circ x$ exists for all $r, s \in [0,1]$, is well-known and we refer for instance to [12]. Besides, we know that $\int_r^s h(x) d^\circ x = H(x_s) - H(x_r)$ for any primitive function H of h. With the help of a Taylor expansion, it is then easy to show (4.14). Finally, the condition (2.7) comes readily in using the identity (2.10), which proves that A is a Lévy area of order 0 and finishes the lemma.

□

The following proposition shows the desired extension.

Proposition 4.2. *With the same notations of Lemma 4.1, we have*

$$\int_a^b f(y_s) d^{A,1} x_s \;=\; \int_a^b f(y_s) d^\circ x_s$$

for any function $f : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^2$.

Proof. Thanks to (4.14), we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^1 f'(y_u) A_{u,u+\varepsilon}(1) du = 0,$$

which entails the required identification.

□

Remark 4.3. We do not know if it is possible to construct a Lévy area

$$\int_a^b f(y_s) d^{A,m} x_s \;=\; \int_a^b f(y_s) d^{NC,m} x_s$$

with the notations of Lemma 4.1, for any function $f : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^{2m}$, in the case $m \geq 2$. An area like

$$A_{rs}(y^q) \;=\; \frac{1}{q+1} \left(\int_r^s y_u^{q+1} \, d^o x_u \; - (x_s - x_r) \int_0^1 (y_r + \theta(y_s - y_r))^{q+1} \, \nu_m(d\theta) \right)$$

for $q \leq m-1$ would be the most natural candidate but unfortunately, only

$$|A_{st}[(y-\xi)^q]| \;\leq\; c|t-s|^{3\alpha}$$

is fulfilled in general, and not (2.8).

Finally, the next corollary show that in Theorem 3.2 our solution-process coincides for $m = 1$ with those given in [6, 13], through a Doss-Sussmann's representation. If we could give a positive answer to the above remark, then the identification with [13] would hold for any $m \geq 2$.

Corollary 4.4. *When $m = 1$ and $\alpha > 1/3$, the unique solution (y, A) to (3.12) can be represented followingly. The function $y : [0, 1] \to \mathbb{R}$ is given by $y_t = u(x_t, a_t)$ where $u : \mathbb{R}^2 \to \mathbb{R}$ is the unique solution to*

$$\frac{\partial u}{\partial x}(x, v) = \sigma(u(x, v)) \text{ and } u(0, v) = v \text{ for any } v \in \mathbb{R}, \tag{4.15}$$

and $a : [0, 1] \to \mathbb{R}$ is the unique solution to

$$\frac{da_t}{dt} = \left\{\frac{\partial u}{\partial a}(x_t, a_t)\right\}^{-1} b \circ u(x_t, a_t) \text{ and } a_0 = y_0. \tag{4.16}$$

The function A is the Lévy area associated to $\gamma = (x, y)$ given by (4.13).

Proof. It is clear that $y \in \mathrm{C}^\alpha$ and we know from Proposition 4.2 that

$$\int_0^t \sigma(y_s) d^{A,1} x_s = \int_0^t \sigma(y_s) d^\circ x_s.$$

The Itô-Stratonovich's formula established in [12], Theorem 4.1.7, shows that

$$u(x_t, a_t) = u(0, a_0) + \int_0^t \frac{\partial u}{\partial x}(x_s, a_s) d^\circ x_s + \int_0^t \frac{\partial u}{\partial a}(x_s, a_s) da_s \tag{4.17}$$

for all $t \in [0, 1]$. Hence, thanks to (4.15) and (4.16),

$$y_t = y_0 + \int_0^t \sigma(y_s) d^\circ x_s + \int_0^t b(y_s) ds = y_0 + \int_0^t b(y_s) ds + \int_0^t \sigma(y_s) d^{A,1} x_s$$

and consequently, (y, A) is the solution to (3.12).

□

Acknowledgement. We thank Denis Feyel for his helpful comments on the first version of this article.

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